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A mixed integer nonlinear programming model for the optimal repair–replacement in the firm

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HIGHLIGHTS

- Proposing an optimal repair–replacement model in firms under technological change.
- Construction of a mixed-integer programming to the new repair–replacement model.
- Investigating the optimality and solvability of the model under some assumptions.
- More discussions on the solvability of the new model in a specific case in economic.
- Providing an approach for solving the specific case of new proposed model.

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ABSTRACT

Optimal repair–replacement problem is an important aspect of economic decision making at the firm and aggregate levels. In this paper, we extend the continuous time optimal replacement model in the firm under technological progress by considering the possibility of repairing/replacing the machines during their lifetime period. In our model, two possible decisions can be recognized by the managers in which the machines are repaired under the efficiency condition or replaced under the availability of technological progress in the firm. As a special case, we restrict the model to the more real case in which all the growth, purchase price and repair cost functions are assumed to be in the exponential form. The solvability of the model in this case is also discussed.

1. Introduction

Based upon economical models, the economic growth can be obtained from the accumulation of three factors: labor, capital and productivity or technological level. Studies on economical models in most countries show that the economic growth is highly dependent on the rate of technological progress, see e.g. Dornbusch et al. (2001). The optimal replacement policy of machines has a key role in companies because it has critical impact on the increase in productivity of the company. Although, the concept of optimal replacement can be analyzed in the both general equilibrium (vintage capital model) and partial equilibrium (machine in firm), in this paper we focus on the latter one.

A great number of papers were written on vintage capital growth model in the 1960s, see e.g. Solow (1962) and Johansen (1959). Solow et al. (1966) constructed one of the first models in general equilibrium replacement decisions and showed that the replacement echoes should vanish in the Solow growth model with vintage capital which probably caused that this area of research in economic stayed silent in 1970s and 1980s, see also Sheshinski (1967). In 1975, Malcomson (1975) proposed a model for optimal replacement of capital equipment in a separate firm and showed that the model can be represented by a nonlinear integral equation with unknown functions in the limit of integration. The analysis of vintage capital growth models has revitalized as an active area of research since the early 1990s as this kind of models allows to address many of the key economic issues such as investment volatility and equipment replacement. Using mathematical literature in economic, Benhabib and Rustichini (1991) showed the existence of periodic solutions in the setup of optimal growth with vintage capital. They have also shown the non-monotonic behavior for vintage models with non-geometric depreciation. Boucekkine et al. (1997) claimed that the result obtained by Solow et al. (1966) comes from the constancy of the saving rate at equilibrium inherent to Solow growth models. They developed an optimization model and showed that replacement echoes do not vanish in the Ramsey vintage capital growth model with linear utility function in which the savings rate is not constant. Investment volatility and

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equipment replacement issues and technology-oriented decisions have been widely studied and analyzed using vintage capital models by Benhabib and Rustichini et al. (1993) and Boucekkine et al. (1998, 1999), Greenwood et al. (1997), Boucekkine et al. (2001) and many others. In Hritonenko and Yatsenko (2008), have shown that the solution of optimal replacement model can be obtained by using a nonlinear Volterra integral equation in which the unknown function appears in both the integrand and the upper limit of the integration. They have also discussed about the existence and uniqueness of the optimal solution of the model in some special cases.

In order to keep and improve the productivity in the firm, managers need to periodically repair its existing machines or buy a new machine with higher technology under some conditions. Due to the nature of technology, it is agreed that the machines with higher technology are more efficient than the older ones. In addition, due to maturity of technological level in some industrial firms, the machines are not rapidly substituted by the new one as in non-high technological firms. In this kind of firms, the manger needs to repair the existing machine to increase its performance for a while until a machine with higher technology could be replaced.

Although, most models of economic growth, like the standard neoclassical growth model, ignore the fact that equipments and machines are maintained and repaired, a Canadian survey, over the period 1961–93, shows that maintenance and repair expenditures in Canada are large enough, see McGratten and Schmitz (1999). Based on this survey, spending on the maintenance and repair of equipment averaged about 6% of the Canadian GDP and over roughly 50% of spending on new equipment. This survey also suggests that the concepts of maintenance and repair and investment are to some degree close substitute for each other. Illuminated by the work presented in McGratten and Schmitz (1999), several research projects have been done regarding the incorporation of maintenance and repair costs in general equilibrium (vintage capital) models of investment and growth, see e.g. Saglam and Veliov (2008), Goetz et al. (2008) and Boucekkine et al. (2009). The activities of maintenance and repair and investment in partial equilibrium are also of interest in the literature, see e.g. McGratten and Schmitz (1999), Rust (1987). In this case, one is faced by a simple possible decision problem and concrete framework to illustrate the idea of maintaining and repairing and replacing equipment in an easy way. The above arguments explains why considering maintenance and repair is important to understand the investment and replacement decisions taken at the general and partial equilibrium levels.

On the other hand, maintenance policy for simple repairable machines has been investigated by many scholars in literatures. Barlow and Hunter (1960) have presented a minimal repair model by considering the same failure rate of the machine as the time of failure. In Brown and Proschan (1983), have focused on an imperfect repair model in which with the probability of p, the repair is perfect and with the probability of $1 - p$, the repair is minimal. Following their consideration, many works have been done by Park (1979), Block et al. (1985), Kijima (1989) and Makis and Jardine (1993). Later in Lam (1988), considered a new stochastic consideration for after repair working times and introduced a so called monotone Geometric Process (GP) model to describe the maintenance problem. Zhang (1994) generalized the Lam's work by providing a bivariate replacement scheme (T, N) under which at the working time T or at the Nth failure, the system is replaced. Many works have been done on the GP model that was proposed by Lam. Zhang (1999) investigated a GP repair model by assuming periodically performing Preventing Repair (PR) strategy for a simple repairable system. He derived the explicit expression of the long-run average cost rate and analyzed the optimal replacement policy. Later, based on Zhang's research, Lam considered a bivariate replacement policy (T, N), and obtained the optimal bivariate policy by minimizing the average cost rate of the simple repairable system. Recently, Wang and Zhang (2006) investigated the repair–replacement problem for a deterioration cold standby repairable system and proposed a bivariate replacement policy (L, N) to optimize the operating process of the system, where L is the interval length between PR, and N is the PR number of the system before it is replaced.

In this paper, we mainly focus on the theory of optimal repair–replacement of machines in the firm in presence of the technological changes under the assumption that the repaired machine can work as good as the new one. We first provide a mixed integer nonlinear optimization model regarding the optimal repair–replacement of machines in the firms which is an extension of the replacement model proposed by Hritonenko and Yatsenko (2009). We then restrict our model to the case in which all the growth, purchase price and repair cost functions are considered to be exponential functions. A procedure for solving the proposed model in the specific case is provided under some suitable conditions. We finally discuss about the optimality conditions and solvability of the solution method by roughly using the approximately similar results in Hritonenko and Yatsenko (2008, 2009) for the optimal replacement model.

The rest of the paper is organized as follows: In Section 2, we give a mixed integer nonlinear optimization model for the optimal repair–replacement problem under technological progress in which the machines are allowed to be repaired under the efficiency conditions. Section 3 is devoted to restate the proposed model for a specific real case in the economical views. A procedure for solving the proposed model in the specific case is provided in Section 4. A short discussion about the optimality conditions of the unconstrained optimization subproblems in the suggested procedure is given in Section 5 which is almost the same as the results obtained in Hritonenko and Yatsenko (2008, 2009) for the optimal replacement model. We end up the paper by providing some concluding remarks in Section 6.

2. The new optimal repair–replacement model

Let us consider a production process over a planning period $[t_0, T]$, for $T < \infty$. In this paper, we generalize the optimal replacement model of Hritonenko and Yatsenko (2009) to the case when the possibility of machine repairing is also allowed for a single-machine in the firms. The model is described as follows:

Suppose that the first machine exists at the beginning of the process time t_0. So, the repair/replace time can be considered as the sequence $t_{k+1} = t_k + d_{k+1}$, where the optimal lifetime sequence $\{d_k\}$ can be either an infinite lifetime, i.e. $k \in \mathbb{N}$, or finite sequence, i.e. $k = 1, \ldots, N$, with an infinite lifetime $d_{N+1} = \infty$, where $N \geq 0$ is given in advance. Here, we focus on the latter case. Our aim in the optimal repair–replacement model under technological progress is to find the optimal policy $\omega^* = \{d_k^* = t_k^* - t_{k-1}^*;\ k = 1, 2, \ldots, N\}$ in which the profit is maximized over the lifetime horizon $[t_0, \infty)$.

From now on, for ease of reference, we assume that the functions $p(t) \geq 0$, $q(t, u) \geq 0$, with $u \geq t$, and $m(t) \geq 0$ are the purchase price, the efficiency and the repair cost functions, respectively. Moreover, due to the role of $q(t, u)$ in the model, it is basically assumed that the function $q(t, u)$ is a decreasing function with respect to u.

From economic point of view, each machine has a performance level so that violating this level may harm the efficiency of the machine in the firm. In the period $[t_i, t_{i+1}]$, the performance level can be described by imposing a lower bound on the machine efficiency (income) in this period as follows:

$$\int_{t_i}^{t_{i+1}} e^{-ru} q(t_i, u) du \geq e^{-rt_i} \beta(t_{i+1} - t_i) \tag{2.1}$$

where $r > 0$ is a given discount rate and $\beta > 0$ is a given efficiency parameter satisfying $e^{-rt_i}q(t_i, t_i) > \beta$. In other words, the machine should be repaired if the inequality (2.1) is violated. Therefore, for given t_i, the machine is started to be repaired whenever t_{i+1} pass the threshold value $\tilde{t}_{i+1} > t_i$, where $\tilde{t}_{i+1}$ is the solution of the following equation:

$$\int_{t_i}^{t_{i+1}} e^{-ru}q(t_i, u)du = e^{-rt_i}\beta(t_{i+1} - t_i). \tag{2.2}$$

The following lemma states that the above equation has a solution.

Lemma 1. *Let $q(t, u)$, for $t < u$, be a continuous function with respect to u. Then, for given t_i, there exists a solution t_{i+1} satisfying (2.2).*

Proof. Due to the economical concept of the machine deterioration, the function $q(t_i, u)$ is a decreasing function with respect to u when the machine age $u - t$ increases. Moreover, we have $q(t_i, t_i) > \beta$, for $t_i < u$. Therefore, there exists $\bar{u} \in [t_i, \infty)$ so that $q(t_i, \bar{u}) = \beta$. Now, using the Mean Theorem for Integrals, there exists $t_{i+1} \in [\bar{u}, \infty)$ so that (2.2) holds. This completes the proof of the lemma. □

In the next section, we will provide an approximate solution for the Eq. (2.2) by using Taylor expansion.

On the other hand, according to the major role of the technology in productivity, it is rational to buy a new machine with higher technology whenever the total income is equal to the price of a new one. Therefore, the existing machine is replaced by a new one at the threshold time $\hat{t}_{i+1} > t_i$, where $\hat{t}_{i+1}$ is the solution of the following equation:

$$\int_{t_i}^{t_{i+1}} e^{-ru}q(t_i, u)du = e^{-rt_i}p(t_i). \tag{2.3}$$

Now, we are in place to state a mathematical formulation for the optimal repair–replacement model. In our proposed model, in each period, a machine is repaired or replaced. Therefore, a mathematical formulation for the optimal repair–replacement model is then described as follows:

$$\begin{aligned} \max_{\omega} \quad & \Phi(t) = \sum_{i=j}^{N} \int_{t_i}^{t_{i+1}} e^{-ru}q(t_i, u)du - \sum_{i=j+1}^{N} e^{-rt_i}h(t_i) \\ \text{s.t.} \quad & \\ & h(t_i) = \begin{cases} m(t_i), & \tilde{t}_{i+1} \le \hat{t}_{i+1} \\ \quad i = 1, 2, \ldots, N \\ p(t_i), & \hat{t}_{i+1} \le \tilde{t}_{i+1} \\ \quad i = 1, 2, \ldots, N \end{cases} \end{aligned} \tag{2.4}$$

where $\omega = \{d_k = t_k - t_{k-1} \mid k = 1, \ldots, N\}$. Using variables $\delta_i \in \{0, 1\}$, we can easily transfer the problem (2.3) to the following Mixed-Integer Nonlinear Programming (MINLP) problem:

$$\begin{aligned} \max_{\omega} \quad & \Phi(t) = \sum_{i=j}^{N} \int_{t_i}^{t_{i+1}} e^{-ru}q(t_i, u)du - \sum_{i=j+1}^{N} e^{-rt_i}h(t_i) \\ \text{s.t.} \quad & \\ & h(t_i) = \delta_i m(t_i) + (1 - \delta_i)p(t_i), \\ & \quad i = 1, 2, \ldots, N \\ & M(\delta_i - 1) \le \tilde{t}_{i+1} - \hat{t}_{i+1} \le M\delta_i \\ & \quad i = 1, 2, \ldots, N, \\ & \delta_i = 0 \text{ or } 1, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{2.5}$$

where the functions $p(t) \ge 0$, $q(t, u) \ge 0$, with $u \ge t$, and $m(t) \ge 0$ are the purchase price, the efficiency and the repair cost functions, respectively. Moreover, $\omega = \{d_k = t_k - t_{k-1} \mid k = 1, \ldots, N\}$ and $M > 0$ is a big enough constant which is used as a standard rule in the literature for reformulating either–or constraints in the mathematical programming problems.

The first and second term in the objective function is the discounted total product output and the last term indicates the discounted total cost of purchased machines or describes the discounted repair cost.

One knows that in the machines lifetime, for given t, the efficiency function $q(t, u)$ decreases with respect to $u > t$ as its physical deterioration rationally increases with respect to $u - t$. Therefore, starting from replacement time, the machine in use should be repaired to increase its efficiency for a while until the manager comes to the point that the machine does not work in an efficient manner anymore, even after repairing. In this case, the machine should be replaced by a new one with higher technology.

In Section 3, we restrict our discussion about the proposed model to a more specific and real case in the context of economic and management sciences.

3. Analysis of a specific case

Let us restrict the model to the more real case which is commonly accepted in the economic and management sciences literatures (Bethuyne, 1998; Grinyer, 1973; Regnier et al., 2004). To be more precise, we impose some assumptions on the functions of the proposed optimization model in order to provide a qualitative analysis of the new optimal repair–replacement model.

Assume that the model is restricted to the case in which the efficiency, the purchase price and the repair cost functions are exponential functions with respect to t, i.e., for $t < u$, we have

$$\begin{aligned} q(t, u) &= q_0 e^{c_b t - c_d(u-t)}, \quad q_0 > 0 \\ p(t) &= p_0 e^{c_p t}, \quad p_0 > 0 \\ m(t) &= m_0 e^{c_m t}, \quad m_0 > 0 \end{aligned} \tag{3.1}$$

where q_0, p_0, and m_0 are the initial values for the efficiency, the purchase price and the repair cost functions at time $t_0 = 0$, respectively. Moreover, c_b stands for the influence of technological progress on the productivity, c_d denotes the impact of age $u - t$ of capital on its efficiency which in fact consists of the deterioration, c_p is the rate of change in the price of new equipment and c_m is considered as the rate of change in the total repair cost of the machine. We further assume that these parameters satisfy the following condition:

$$c_b + c_p + c_m + c_d > 0. \tag{3.2}$$

As mentioned above, efficiency condition expressed in (2.2), in exponential case will result in the following theorem:

Theorem 1. *Assume that the efficiency function $q(t, u)$ is given by (3.1). Then, for given β and t_i, an approximation of the solution $\tilde{t}_{i+1}$ of (2.2) is provided by*

$$\tilde{t}_{i+1} \simeq t_i + \frac{2}{r + c_d}\left(1 - \frac{\beta}{q_0}e^{-c_b t_i}\right). \tag{3.3}$$

Proof. Substituting (3.1) in (2.2) leads us to the following equation:

$$\begin{aligned} e^{-rt_i}\beta(t_{i+1} - t_i) &= \int_{t_i}^{t_{i+1}} e^{-ru}q_0 e^{c_b t_i - c_d(u-t_i)}du \\ &= \frac{q_0}{-(r + c_d)}\left(e^{-rt_{i+1} + c_b t_i - c_d(t_{i+1}-t_i)} - e^{(c_b - r)t_i}\right) \\ &= \frac{q_0 e^{(c_b - r)t_i}}{-(r + c_d)}\left(e^{-(r+c_d)(t_{i+1}-t_i)} - 1\right). \end{aligned}$$

Thus, we have

$$\frac{q_0}{-(r + c_d)}\left(e^{-(r+c_d)(t_{i+1}-t_i)} - 1\right) = \beta(t_{i+1} - t_i)e^{-c_b t_i}.$$

Now, by using the second order Taylor expansion, we obtain:

$$\frac{q_0}{-(r+c_d)}\left(\frac{(r+c_d)^2}{2}(t_{i+1}-t_i)^2-(r+c_d)(t_{i+1}-t_i)\right)$$
$$\simeq \beta(t_{i+1}-t_i)e^{-c_b t_i}.$$

Simplifying this equation implies that:

$$\left(\frac{\beta}{-q_0}e^{-c_b t_i}+1\right)\frac{2}{r+c_d}\simeq \tilde{t}_{i+1}-t_i$$

which completes the proof. □

On the other hand, the technological progress plays a key role in our decision for repairing or replacing the machines. Therefore, the machine should be replaced by a new one soon after the total income is equal to the price of a new machine with higher technology. In this case, Eq. (2.3) should be satisfied. The following theorem provides an approximate solution for this equation.

Theorem 2. *Let $q(t,u)$ and $p(t)$ be the efficiency and purchase price functions as given in (3.1). Then, an approximation of the solution $\hat{t}_{i+1}$ of (2.3) is provided by*

$$\hat{t}_{i+1}\simeq t_i+\frac{p_0}{q_0}\left(1+(c_p-c_b)t_i\right). \quad (3.4)$$

Proof. Substituting (3.1) in (2.3) leads us to the following equations:

$$\begin{aligned} p_0e^{(c_p-r)t_i} &= \int_{t_i}^{t_{i+1}} e^{-ru}q_0e^{c_bt_i-c_d(u-t_i)}du \\ &= \left(\frac{q_0}{-(r+c_d)}\right)\left(e^{-rt_{i+1}+c_bt_i-c_d(t_{i+1}-t_i)}-e^{-rt_i+c_bt_i}\right) \\ &= \left(\frac{q_0e^{(c_b-r)t_i}}{-(r+c_d)}\right)\left(e^{-(r+c_d)(t_{i+1}-t_i)}-1\right). \end{aligned}$$

Thus, we have

$$e^{(c_p-c_b)t_i}=\left(\frac{q_0}{-(r+c_d)p_0}\right)\left(e^{-(r+c_d)(t_{i+1}-t_i)}-1\right).$$

By using the first order Taylor expansion for the exponential functions in the above equation, we obtain:

$$1+(c_p-c_b)t_i\simeq\left(\frac{q_0}{-(r+c_d)p_0}\right)$$
$$\times\left[1-(r+c_d)(t_{i+1}-t_i)-1\right].$$

Simplifying this equation implies that:

$$1+(c_p-c_b)t_i\simeq\frac{q_0}{p_0}(t_{i+1}-t_i).$$

This completes the proof of the theorem. □

From now on, we set the approximations of $\tilde{t}_{i+1}$ and $\hat{t}_{i+1}$ by

$$\bar{\tilde{t}}_{i+1}=t_i+\frac{2}{r+c_d}\left(1-\frac{\beta}{q_0}e^{-c_bt_i}\right) \quad (3.5)$$

and

$$\bar{\hat{t}}_{i+1}=t_i+\frac{p_0}{q_0}\left(1+(c_p-c_b)t_i\right) \quad (3.6)$$

respectively. In the next section, we are going to discuss about an approach for solving problem (2.5) using these approximations.

4. How to solve the model

In this section, we discuss about an approach for solving a variant of problem (2.5) in specific case in which the parameters $\tilde{t}_{i+1}$ and $\hat{t}_{i+1}$ are replaced by their approximate values $\bar{\tilde{t}}_{i+1}$ and $\bar{\hat{t}}_{i+1}$, respectively. Our proposed procedure works iteratively as follows:

For given t_0, at the i-th iteration, let t_i be known. In order to compute t_{i+1}, we first calculate the parameters $\bar{\tilde{t}}_{i+1}$ and $\bar{\hat{t}}_{i+1}$ by (3.5) and (3.6), respectively. Two possible cases may happen in the feasible region of the problem (2.5), which are:

Case 1: $\bar{\tilde{t}}_{i+1}\geq\bar{\hat{t}}_{i+1}$.
In this case, we have $m(t_{i+1})\geq p(t_{i+1})$. Simply speaking, this means that the repair cost is more than the purchase price of a new high-tech device and therefore the machine should be replaced by a new one. Therefore, in the problem (2.5), we have $\delta_i=0$ which implies that $h(t_{i+1})=p(t_{i+1})$. This leads the problem (2.5) to the following unconstrained reformulation:

$$\max_{\omega}\quad \Phi(t)=\sum_{j=i}^{N}\int_{t_j}^{t_{j+1}}e^{-ru}q(t_j,u)du-\sum_{j=i+1}^{N}e^{-rt_j}p(t_j). \quad (4.1)$$

Case 2: $\bar{\tilde{t}}_{i+1}<\bar{\hat{t}}_{i+1}$.
In this case, we have $m(t_{i+1})<p(t_{i+1})$ which means that the purchase price is beyond the repair cost and it is rational to decide for repairing the existing machine. This leads the problem (2.5) to choose $\delta_i=1$ and therefore $h(t_{i+1})=m(t_{i+1})$. Thus, the problem (2.5) can be converted to the following unconstrained reformulation:

$$\max_{\omega}\quad \Phi(t)=\sum_{j=i}^{N}\int_{t_j}^{t_{j+1}}e^{-ru}q(t_j,u)du-\sum_{j=i+1}^{N}e^{-rt_j}m(t_j). \quad (4.2)$$

After solving the problem (4.1) or (4.2), an optimal sequence $\omega^*_{i+1}=\{d^*_k=t^*_k-t^*_{k-1}\mid k=i+1,\ldots,N\}$ is obtained. Letting $t_{i+1}=t^*_{i+1}$, this procedure can be repeated for the iteration $i+1$.

Remark. It is worth to mention that when the repair cost is equal to the purchase price, regarding the higher efficiency of a new machine with higher technology, we prefer to buy a new machine instead of repairing the existing one.

As the problems (4.1) and (4.2) are similar to the structure of the optimal replacement model in Hritonenko and Yatsenko (2009), in both cases, solvability and optimality conditions of these two problems are analogous to that of proposed in Hritonenko and Yatsenko (2009). In the next section, we just provide the results regarding the solvability and optimality conditions of the problems (4.1) and (4.2) without any proof. We refer the interested reader to Hritonenko and Yatsenko (2009) in order to construct an analogous proof for these results.

5. Optimality discussion

As the problems (4.1) and (4.2) have a similar structure, in this section, we just briefly discuss about the solvability and optimality conditions of the problem (4.1) in general and in specific cases based on the selection of the parameters c_p, c_b and c_m. The results are taken from the similar results of the optimal replacement models proposed by Hritonenko (2005) and Hritonenko and Yatsenko (2008, 2009).

The following lemma states the optimality conditions for the problem (4.1).

Lemma 2. *Let $q(t,u)$, $p(t)$ and $m(t)$ are continuously differentiable functions with respect to t and continuous functions with respect to $u(>t)$. Assume that an optimal policy $\omega^*=\{d^*_k=t^*_k-t^*_{k-1};k=$*

$1, 2, \ldots, N\}$ *exists for the problem* (4.1). *Then, for all* $i = 1, 2, \ldots, N$, *the lifetime* d_i^* *satisfies the following equation:*

$$\int_{t_i}^{t_{i+1}} e^{-ru} \frac{\partial q(t_i, u)}{\partial t_i} du + e^{-rt_i}(rp(t_i) - p'(t_i)) + e^{-rt_i}[q(t_{i-1}, t_i) - q(t_i, t_i)] = 0. \quad (5.1)$$

Proof. See Hritonenko and Yatsenko (2009). □

In the specific case, substituting (3.1) in (5.1) leads us to the following equation:

$$\frac{q_0(c_b + c_d)}{r + c_d}\left(1 - e^{-(r+c_d)d_{i+1}}\right) + q_0\left(e^{-(c_b+c_d)d_i} - 1\right) + p_0 e^{(c_p-c_b)t_i}(r - c_p) = 0. \quad (5.2)$$

Now, we discuss about the solvability of (5.2) based on the various choices of the parameters c_b, c_p and c_m. This issue is done according to the forthcoming results.

Theorem 3. *Let* $c_b = c_p = c_m = c$. *Then, there exists an optimal policy* $\omega^* = \{d_k^* = d > 0; k = 1, 2, \ldots, N\}$ *for the Eq.* (5.2), *where d is determined by solving the following nonlinear equation:*

$$c_d e^{-(c+c_d)d} - (c + c_d)e^{-c_d d} - \left(\frac{p_0}{q_0}c_d - 1\right)c = 0. \quad (5.3)$$

In the case $r \ll 1$, *an approximate solution for* (5.2) *is given by:*

$$d = \sqrt{\frac{\frac{2p_0}{q_0}}{c + c_d}} + o(r). \quad (5.4)$$

Proof. A proof can be found in Hritonenko and Yatsenko (2008, 2009). □

Note that under the condition $c_b = c_p = c_m = c$, the optimal repair–replacement will be a constant lifetime. As mentioned before, in this case we prefer to replace the machine with a new high-tech one instead of repairing it.

Now, we continue the specific choices of parameters in problem (5.2) by noting that the case $c_b \neq c_p \neq c_m$ is more complicated to discuss. Therefore, we just consider the solvability of the system for one more real case in which we have $c_b = c_p > c_m$:

Theorem 4. *Let* $c_b > c_p = c_m$. *Then, the nonlinear equation* (5.2) *has a unique optimal policy* $\omega^* = \{d_k^*; k = 1, 2, \ldots, N\}$.

Proof. For proof, see the Hritonenko and Yatsenko (2008, 2009). □

Let us look at the nonlinear equation (5.2) from another point of view. Let $x(t)$ be defined as $x(t) = t - d(t)$. Then, from $d_{i+1} = t_{i+1} - t_i$, we have $t_i = x(t_{i+1})$, which implies that $t_{i+1} = x^{-1}(t_i)$. On the other hand, from $d(t_i) = t_i - t_{i-1}$, we have $x(t_i) = t_{i-1}$. Therefore, by rewriting Eq. (5.2) according to the function $x(t)$ and substituting t_i by t and multiplying $-q_0 e^{(-r+c_b)t}$ in the both sides of equation, we obtain:

$$\frac{q_0(c_b + c_d)}{r + c_d}\left(e^{-rx^{-1}(t) - c_d(x^{-1}(t)-t) + c_b t} - e^{-rt + c_b t}\right) - e^{-rt}\left[q_0 e^{c_b x(t) - c_d(x(t)-t)} - q_0 e^{c_b t}\right] = p_0 e^{(c_p - r)t}(c_p - r).$$

This equation is exactly the derivative of the following Volterra integral equation with respect to t:

$$\int_t^{x^{-1}(t)} e^{-ru}\left[q_0 e^{c_b x(u) - c_d(x(u)-u)} - q_0 e^{c_b t - c_d(u-t)}\right] du = p_0 e^{c_p t}, \quad (5.5)$$

where $x^{-1}(t)$ stands for the inverse function of $x(t)$. This concept are fully discussed on Hritonenko and Yatsenko (2008, 2009).

The integral equation (5.5) is a kind of nonlinear Volterra integral equation in which the unknown function appears in the upper bound of the integration and the integrand simultaneously. This makes the problem a little bit complicate. Using some mathematical transformations, it can be changed to a delay-type integral equations. An approach for solving this kind of integral equation was proposed by Hritonenko and Yatsenko in (2009).

From practical point of view, the optimal repair–replacement model proposed in this paper may have more difficulty for solving in general. In real world problems, the coefficients, c_b, c_d, c_p and c_m in our model are usually restricted to be some specific forms, i.e. the exponential forms which are more common in the real applications. Moreover, in presence of technological level, the rates c_b, c_m and c_p are considered to be the same due to their economical interpretations in the firms.

6. Conclusion

In this paper, the optimal replacement model in the firms with attention to the technological progress is extended to the case in which the cost of repair is also considered. An approach for solving the proposed model is presented. We also provide some discussions regarding the optimality conditions and solvability of the model under some standard assumptions in general and in an specific case in which the functions in the model are restricted to be exponential functions.

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